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A NOTE ON CONTIGUITY AND HELLINGER DISTANCE

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A note on contiguity and Hellinger distance

by

J. Oosterhoff & W.R. van Zwet

ABSTRACT

Two sequences of product probability measures $\{\prod_{i=1}^n P_{ni}\}$ and $\{\prod_{i=1}^n Q_{ni}\}$ are considered. Necessary and sufficient conditions in terms of the marginal distributions P_{ni} and Q_{ni} are derived for the contiguity of the sequences. Boundedness for $n \rightarrow \infty$ of the sum of squares of the Hellinger distances of the marginals is one of the conditions. By strengthening these conditions one obtains sufficient and (almost) necessary conditions for the asymptotic normality of the log likelihood ratio statistic.

KEY WORDS & PHRASES: *asymptotic normality, contiguity, Hellinger distance, log likelihood ratio*

1. INTRODUCTION

For $n = 1, 2, \dots$ let $(X_{n1}, A_{n1}), \dots, (X_{nn}, A_{nn})$ be arbitrary measurable spaces. Let P_{ni} and Q_{ni} be probability measures defined on (X_{ni}, A_{ni}) , $i = 1, \dots, n$; $n = 1, 2, \dots$, and let $P_n^{(n)} = \prod_{i=1}^n P_{ni}$ and $Q_n^{(n)} = \prod_{i=1}^n Q_{ni}$ ($n=1, 2, \dots$) denote the product probability measures. For each i and n let X_{ni} be the identity map from X_{ni} onto X_{ni} . Then P_{ni} and Q_{ni} represent the two possible distributions of the random element X_{ni} as well as the probability measures of the underlying probability space. Obviously X_{n1}, \dots, X_{nn} are independent under both $P_n^{(n)}$ and $Q_n^{(n)}$ ($n=1, 2, \dots$).

The sequence $\{Q_n^{(n)}\}$ is said to be contiguous with respect to the sequence $\{P_n^{(n)}\}$ if $\lim_{n \rightarrow \infty} P_n^{(n)}(A_n) = 0$ implies $\lim_{n \rightarrow \infty} Q_n^{(n)}(A_n) = 0$ for any sequence of measurable sets A_n . This one-sided contiguity notion is denoted by $\{Q_n^{(n)}\} \triangleleft \{P_n^{(n)}\}$ (the notation is due to H. WITTING & G. NÖLLE [7]). The sequences $\{P_n^{(n)}\}$ and $\{Q_n^{(n)}\}$ are said to be contiguous with respect to each other if both $\{Q_n^{(n)}\} \triangleleft \{P_n^{(n)}\}$ and $\{P_n^{(n)}\} \triangleleft \{Q_n^{(n)}\}$. This two-sided contiguity concept we denote by $\{P_n^{(n)}\} \triangleleft\!\!\!\triangleleft \{Q_n^{(n)}\}$.

The main purpose of this note is to characterize contiguity of product probability measures in terms of their marginals. To this end we introduce the Hellinger distance $H(P, Q)$ between two probability measures P and Q on the same σ -field, defined by

$$(1.1) \quad H(P, Q) = \left\{ \int (p^{\frac{1}{2}} - q^{\frac{1}{2}})^2 d\mu \right\}^{\frac{1}{2}} = \left\{ 2 - 2 \int p^{\frac{1}{2}} q^{\frac{1}{2}} d\mu \right\}^{\frac{1}{2}},$$

where $p = dP/d\mu$, $q = dQ/d\mu$ and μ is any σ -finite measure dominating $P+Q$. This metric is independent of the choice of μ and satisfies $0 \leq H(P, Q) \leq 2^{\frac{1}{2}}$. Defining the total variation distance of P and Q by

$$(1.2) \quad \|P - Q\| = \sup |P(A) - Q(A)|,$$

where the supremum is taken over all measurable sets A , we have the following inequalities (LECAM [4])

$$(1.3) \quad \frac{1}{2} H^2(P, Q) \leq \|P - Q\| \leq H(P, Q).$$

The Hellinger distances of the product measures and of their marginals are connected by the relationship

$$(1.4) \quad H^2(P_n^{(n)}, Q_n^{(n)}) = 2 - 2 \prod_{i=1}^n \{1 - \frac{1}{2} H^2(P_{ni}, Q_{ni})\}.$$

For further reference we first mention two easy results, viz.

$$(1.5) \quad \sum_{i=1}^n H^2(P_{ni}, Q_{ni}) = o(1) \quad \text{for } n \rightarrow \infty \Rightarrow \{P_n^{(n)}\} \triangleleft \{Q_n^{(n)}\},$$

and

$$(1.6) \quad \{Q_n^{(n)}\} \triangleleft \{P_n^{(n)}\} \Rightarrow \sum_{i=1}^n H^2(P_{ni}, Q_{ni}) = o(1) \quad \text{for } n \rightarrow \infty.$$

The proof of (1.5) is an immediate consequence of the string of implications

$$\sum_{i=1}^n H^2(P_{ni}, Q_{ni}) = o(1) \Rightarrow \sum_{i=1}^n \log\{1 - \frac{1}{2} H^2(P_{ni}, Q_{ni})\} = o(1) \Rightarrow \\ \Rightarrow H^2(P_n^{(n)}, Q_n^{(n)}) = o(1) \Rightarrow \|P_n^{(n)} - Q_n^{(n)}\| = o(1) \Rightarrow \{P_n^{(n)}\} \triangleleft \{Q_n^{(n)}\}.$$

To prove (1.6) suppose that $\limsup_{n \rightarrow \infty} H(P_n^{(n)}, Q_n^{(n)}) = 2^{\frac{1}{2}}$. Then by (1.3)

$$\limsup_{n \rightarrow \infty} \|P_n^{(n)} - Q_n^{(n)}\| = 1 \text{ in contradiction to } \{Q_n^{(n)}\} \triangleleft \{P_n^{(n)}\}.$$

$$\text{Thus } \limsup_{n \rightarrow \infty} H^2(P_n^{(n)}, Q_n^{(n)}) < 2, \text{ therefore } \liminf_{n \rightarrow \infty} \prod_{i=1}^n \{1 - \frac{1}{2} H^2(P_{ni}, Q_{ni})\} > 0$$

and hence $\limsup_{n \rightarrow \infty} \sum_{i=1}^n H^2(P_{ni}, Q_{ni}) < \infty$ and the proof is complete.

It can be shown by counterexamples that in (1.5) the condition cannot be weakened to $\sum_{i=1}^n H^2(P_{ni}, Q_{ni}) = o(1)$, and that in (1.6) the conclusion cannot be strengthened to $\sum_{i=1}^n H^2(P_{ni}, Q_{ni}) = o(1)$, for $n \rightarrow \infty$. Hence there remains a gap between the sufficient condition and the necessary condition for contiguity in (1.5) and (1.6) respectively. In section 2 we obtain conditions which are both sufficient and necessary for contiguity of the product measures by adding another condition to $\sum_{i=1}^n H^2(P_{ni}, Q_{ni}) = o(1)$.

In many applications asymptotic normality of the log likelihood ratio statistic Λ_n (see (3.1)) plays an important part. Since $L(\Lambda_n | P_n^{(n)}) \xrightarrow{w} N(-\frac{1}{2}\sigma^2; \sigma^2)$ implies $\{P_n^{(n)}\} \triangleleft \{Q_n^{(n)}\}$ (cf. HÁJEK & ŠIDÁK [1], LECAM [2], [3], [4], ROUSSAS [6]), we have to impose stronger conditions on the marginals P_{ni} and Q_{ni} to ensure the asymptotic normality of Λ_n . Some sufficient (and almost necessary) conditions for the asymptotic normality of Λ_n , which are clearly stronger than those in section 2, are given in section 3. These conditions are closely related to some earlier results of LECAM [3], [4].

2. CONTIGUITY OF PRODUCT MEASURES

We begin by noting the following useful implication:

$$(2.1) \quad \{Q_n^{(n)}\} \triangleleft \{P_n^{(n)}\} \Rightarrow \\ \Rightarrow [\lim_{n \rightarrow \infty} \sum_{i=1}^n P_{ni}(A_{ni}) = 0 \Rightarrow \lim_{n \rightarrow \infty} \sum_{i=1}^n Q_{ni}(A_{ni}) = 0]$$

for any collection of measurable sets A_{ni} . For suppose $\lim_{n \rightarrow \infty} \sum_{i=1}^n P_{ni}(A_{ni}) = 0$. Then $\lim_{n \rightarrow \infty} P_n^{(n)}(\bigcup_{i=1}^n A_{ni}) = 0$, hence by contiguity $\lim_{n \rightarrow \infty} Q_n^{(n)}(\bigcup_{i=1}^n A_{ni}) = 1 - \lim_{n \rightarrow \infty} \prod_{i=1}^n (1 - Q_{ni}(A_{ni})) = 0$ and therefore $\lim_{n \rightarrow \infty} \sum_{i=1}^n Q_{ni}(A_{ni}) = 0$.

Now let μ_{ni} be a σ -finite measure on (X_{ni}, A_{ni}) dominating $P_{ni} + Q_{ni}$ and write $p_{ni} = dP_{ni}/d\mu_{ni}$ and $q_{ni} = dQ_{ni}/d\mu_{ni}$ ($i=1, \dots, n; n=1, 2, \dots$). The main result of this section is

THEOREM 1. $\{Q_n^{(n)}\} \triangleleft \{P_n^{(n)}\}$ iff

$$(2.2) \quad \limsup_{n \rightarrow \infty} \sum_{i=1}^n H^2(P_{ni}, Q_{ni}) < \infty$$

and

$$(2.3) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n Q_{ni}(q_{ni}(X_{ni})/p_{ni}(X_{ni}) \geq c_n) = 0 \quad \text{whenever } c_n \rightarrow \infty.$$

PROOF. First assume that (2.2) and (2.3) are satisfied. Write

$$L_{ni} = q_{ni}(X_{ni})/p_{ni}(X_{ni}), \quad i = 1, \dots, n; n = 1, 2, \dots,$$

and consider $\prod_{i=1}^n L_{ni}$. It is easily shown (cf. LECAM [4], ROUSSAS [6]) that $\{Q_n^{(n)}\} \triangleleft \{P_n^{(n)}\}$ is equivalent to tightness of the sequence of distributions $\{L(\prod_{i=1}^n L_{ni} | Q_n^{(n)}); n=1, 2, \dots\}$. The tightness of this set of distributions can also be expressed in the more convenient form

$$(2.4) \quad \lim_{n \rightarrow \infty} Q_n^{(n)}(\prod_{i=1}^n L_{ni} \geq k_n) = 0 \quad \text{whenever } k_n \rightarrow \infty.$$

Hence we have to prove (2.4). Let $0 < k_n \rightarrow \infty$. Let $0 < c_n \rightarrow \infty$ be real numbers to be chosen in the sequel. If 1_A denotes the indicator function of the set A , we have by (2.3) and Markov's inequality for $n \rightarrow \infty$

$$\begin{aligned}
& Q_n^{(n)}(\Pi_{i=1}^n L_{ni} \geq k_n) \leq \\
& \leq Q_n^{(n)}(\Pi_{i=1}^n L_{ni} \geq k_n \wedge L_{ni} < c_n \text{ for } i=1, \dots, n) + Q_n^{(n)}(\cup_{i=1}^n \{L_{ni} \geq c_n\}) \leq \\
& \leq Q_n^{(n)}(\Pi_{i=1}^n L_{ni}^{\frac{1}{2}} 1_{(0, c_n)}(L_{ni}) \geq k_n^{\frac{1}{2}}) + \sum_{i=1}^n Q_{ni}(L_{ni} \geq c_n) \leq \\
& \leq k_n^{-\frac{1}{2}} \Pi_{i=1}^n \int_{q_{ni} < c_n p_{ni}} q_{ni}^{\frac{3}{2}} p_{ni}^{-\frac{1}{2}} d\mu_{ni} + o(1).
\end{aligned}$$

Since for all $c_n \geq 1$

$$\begin{aligned}
& \int_{q_{ni} < c_n p_{ni}} q_{ni}^{\frac{3}{2}} p_{ni}^{-\frac{1}{2}} d\mu_{ni} \leq \\
& \leq \int_{q_{ni} < c_n p_{ni}} q_{ni} d\mu_{ni} + \int_{q_{ni} < c_n p_{ni}} q_{ni} p_{ni}^{-\frac{1}{2}} (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}}) d\mu_{ni} \leq \\
& \leq 1 + \int_{q_{ni} < c_n p_{ni}} q_{ni}^{\frac{1}{2}} p_{ni}^{-\frac{1}{2}} (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}})^2 d\mu_{ni} + \int_{q_{ni} < c_n p_{ni}} q_{ni}^{\frac{1}{2}} (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}}) d\mu_{ni} \leq \\
& \leq 1 + c_n^{\frac{1}{2}} \int (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}})^2 d\mu_{ni} + 1 - \int q_{ni}^{\frac{1}{2}} p_{ni}^{\frac{1}{2}} d\mu_{ni} - \int_{q_{ni} \geq c_n p_{ni}} q_{ni}^{\frac{1}{2}} (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}}) d\mu_{ni} \leq \\
& \leq 1 + (c_n^{\frac{1}{2}} + \frac{1}{2}) H^2(P_{ni}, Q_{ni}),
\end{aligned}$$

it follows that

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} Q_n^{(n)}(\Pi_{i=1}^n L_{ni} \geq k_n) \leq \\
& \leq \limsup_{n \rightarrow \infty} k_n^{-\frac{1}{2}} \Pi_{i=1}^n \{1 + (c_n^{\frac{1}{2}} + \frac{1}{2}) H^2(P_{ni}, Q_{ni})\} \leq \\
& \leq \limsup_{n \rightarrow \infty} k_n^{-\frac{1}{2}} \exp\{(c_n^{\frac{1}{2}} + \frac{1}{2}) \sum_{i=1}^n H^2(P_{ni}, Q_{ni})\}.
\end{aligned}$$

Choosing c_n in such a way that $c_n = o((\log k_n)^2)$ for $n \rightarrow \infty$, (2.2) implies $Q_n^{(n)}(\prod_{i=1}^n L_{ni} \geq k_n) = o(1)$ for $n \rightarrow \infty$ and (2.4) is established.

Conversely, suppose that $\{Q_n^{(n)}\} \triangleleft \{P_n^{(n)}\}$. Since (1.6) implies that (2.2) is satisfied, it remains to prove (2.3). Let $0 < c_n \rightarrow \infty$ and consider the inequality, valid for $c_n \geq 4$,

$$\begin{aligned} \sum_{i=1}^n \int_{q_{ni} \geq c_n p_{ni}} p_{ni} d\mu_{ni} &\leq c_n^{-\frac{1}{2}} \sum_{i=1}^n \int_{q_{ni} \geq c_n p_{ni}} p_{ni}^{\frac{1}{2}} q_{ni}^{\frac{1}{2}} d\mu_{ni} = \\ &= c_n^{-\frac{1}{2}} \left\{ \sum_{i=1}^n \int_{q_{ni} \geq c_n p_{ni}} p_{ni}^{\frac{1}{2}} (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}}) d\mu_{ni} + \sum_{i=1}^n \int_{q_{ni} \geq c_n p_{ni}} p_{ni} d\mu_{ni} \right\} \leq \\ &\leq c_n^{-\frac{1}{2}} \left\{ \sum_{i=1}^n \int_{q_{ni} \geq c_n p_{ni}} (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}})^2 d\mu_{ni} + \sum_{i=1}^n \int_{q_{ni} \geq c_n p_{ni}} p_{ni} d\mu_{ni} \right\} \leq \\ &\leq c_n^{-\frac{1}{2}} \left\{ \sum_{i=1}^n H^2(P_{ni}, Q_{ni}) + \sum_{i=1}^n \int_{q_{ni} \geq c_n p_{ni}} p_{ni} d\mu_{ni} \right\}. \end{aligned}$$

Since by (2.2) $c_n^{-\frac{1}{2}} \sum_{i=1}^n H^2(P_{ni}, Q_{ni}) \rightarrow 0$ for $n \rightarrow \infty$, it follows that

$\lim_{n \rightarrow \infty} \sum_{i=1}^n P_n(L_{ni} \geq c_n) = 0$. Hence (2.1) implies that

$\lim_{n \rightarrow \infty} \sum_{i=1}^n Q_{ni}(L_{ni} \geq c_n) = 0$ and the proof of the theorem is complete. \square

COROLLARY 1. $\{P_n^{(n)}\} \triangleleft \{Q_n^{(n)}\}$ iff (2.2) and (2.3) are satisfied and

$$(2.5) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n P_{ni}(p_{ni}(X_{ni})/q_{ni}(X_{ni}) \geq c_n) = 0 \quad \text{whenever } c_n \rightarrow \infty.$$

In connection with contiguity Hellinger distance seems to be a more appropriate metric than total variation distance. Note that from (1.3) and (1.6) we immediately obtain the implication

$$(2.6) \quad \{Q_n^{(n)}\} \triangleleft \{P_n^{(n)}\} \Rightarrow \sum_{i=1}^n \|P_{ni} - Q_{ni}\|^2 = o(1) \quad \text{for } n \rightarrow \infty,$$

where again the order term cannot be strengthened to $o(1)$. However, $\sum_{i=1}^n \|P_{ni} - Q_{ni}\|^2 = o(1)$ is too weak a condition to replace (2.2) in theo-

rem 1. On the other hand we cannot strengthen this condition to $\sum_{i=1}^n \|P_{ni} - Q_{ni}\|^r = O(1)$ for some $r < 2$, since $\{Q_n^{(n)}\} \triangleleft \{P_n^{(n)}\}$ does not necessarily imply $\sum_{i=1}^n \|P_{ni} - Q_{ni}\|^r = O(1)$ for any positive $r < 2$. The following example serves to illustrate these points.

EXAMPLE. Let μ_{ni} denote Lebesgue measure on $(0,1)$, let $p_{ni} = 1_{(0,1)}$ and let $q_{ni} = (1+n^{-\frac{1}{2}}) 1_{(0,1-n^{-\frac{1}{2}})} + n^{-\frac{1}{2}} 1_{[1-n^{-\frac{1}{2}},1)}$, $i = 1, \dots, n$; $n = 1, 2, \dots$. Then $\sum_{i=1}^n \|P_{ni} - Q_{ni}\|^2 = (1-n^{-\frac{1}{2}})^2 \leq 1$ and (2.3) is trivially satisfied since q_{ni}/p_{ni} is uniformly bounded. But $\{Q_n^{(n)}\} \triangleleft \{P_n^{(n)}\}$ does not hold because $\sum_{i=1}^n H^2(P_{ni}, Q_{ni}) = 2n\{1 - \int q_{ni}^{\frac{1}{2}} d\mu_{ni}\} = 2n\{1 - (1+n^{-\frac{1}{2}})^{\frac{1}{2}} (1-n^{-\frac{1}{2}}) - n^{-\frac{3}{4}}\} = n^{\frac{1}{2}}(1+o(1))$ for $n \rightarrow \infty$.

Taking $q_{ni} = (1+n^{-\frac{1}{2}}) 1_{(0, \frac{1}{2})} + (1-n^{-\frac{1}{2}}) 1_{[\frac{1}{2}, 1)}$ for all i and n , we have $\{Q_n^{(n)}\} \triangleleft \{P_n^{(n)}\}$ since (2.3) is satisfied and $\sum_{i=1}^n H^2(P_{ni}, Q_{ni}) = 2n\{1 - \frac{1}{2}(1+n^{-\frac{1}{2}})^{\frac{1}{2}} - \frac{1}{2}(1-n^{-\frac{1}{2}})^{\frac{1}{2}}\} = \frac{1}{4} + o(1)$ for $n \rightarrow \infty$. However, in this case $\sum_{i=1}^n \|P_{ni} - Q_{ni}\|^r = n(\frac{1}{2}n^{-\frac{1}{2}})^r \rightarrow \infty$ for $n \rightarrow \infty$ if $r < 2$.

3. ASYMPTOTIC NORMALITY OF Λ_n

Define

$$(3.1) \quad \Lambda_n = \sum_{i=1}^n \log\{q_{ni}(X_{ni})/p_{ni}(X_{ni})\}, \quad n = 1, 2, \dots$$

Note that, with probability one, Λ_n is well-defined under $P_n^{(n)}$, although Λ_n may assume the value $-\infty$ with positive probability under $P_n^{(n)}$.

In our search for necessary and sufficient conditions for the weak convergence $L(\Lambda_n | P_n^{(n)}) \rightarrow_w N(-\frac{1}{2}\sigma^2; \sigma^2)$ in terms of the marginal distributions of the X_{ni} we shall confine ourselves to the case where the summands in (3.1) satisfy the traditional u.a.n. condition (cf. LOÈVE [5]).

THEOREM 2. For any $\sigma \geq 0$

$$(3.2) \quad L(\Lambda_n | P_n^{(n)}) \rightarrow_w N(-\frac{1}{2}\sigma^2; \sigma^2)$$

and

$$(3.3) \quad \lim_{n \rightarrow \infty} \max_{1 \leq i \leq n} P_{ni}(|\log\{q_{ni}(X_{ni})/p_{ni}(X_{ni})\}| \geq \epsilon) = 0$$

for every $\epsilon > 0$ iff for every $\epsilon > 0$

$$(3.4) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n H^2(P_{ni}, Q_{ni}) = \frac{1}{4}\sigma^2,$$

$$(3.5) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n Q_{ni}(q_{ni}(X_{ni})/p_{ni}(X_{ni}) \geq 1+\epsilon) = 0,$$

$$(3.6) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n P_{ni}(p_{ni}(X_{ni})/q_{ni}(X_{ni}) \geq 1+\epsilon) = 0,$$

or equivalently, iff (3.4) holds and for every $\epsilon > 0$

$$(3.7) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_{|q_{ni}-p_{ni}| \geq \epsilon p_{ni}} (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}})^2 d\mu_{ni} = 0.$$

PROOF. To simplify the notation we write $r_{ni} = q_{ni}/p_{ni}$. We first show that (3.5) and (3.6) are equivalent to (3.7). From

$$\begin{aligned} & \sum_{i=1}^n \int_{|q_{ni}-p_{ni}| \geq \epsilon p_{ni}} (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}})^2 d\mu_{ni} = \\ & = \sum_{i=1}^n \left\{ \int_{r_{ni} \geq 1+\epsilon} q_{ni}(1-r_{ni}^{-\frac{1}{2}})^2 d\mu_{ni} + \int_{r_{ni} \leq 1-\epsilon} p_{ni}(1-r_{ni}^{\frac{1}{2}})^2 d\mu_{ni} \right\} \end{aligned}$$

we obtain the double inequality

$$\begin{aligned} & \{1 - (1+\epsilon)^{-\frac{1}{2}}\}^2 \sum_{i=1}^n Q_{ni}(r_{ni}(X_{ni}) \geq 1+\epsilon) + \\ & + \{1 - (1-\epsilon)^{\frac{1}{2}}\}^2 \sum_{i=1}^n P_{ni}(r_{ni}^{-1}(X_{ni}) \geq (1-\epsilon)^{-1}) \leq \\ & \leq \sum_{i=1}^n \int_{|q_{ni}-p_{ni}| \geq \epsilon p_{ni}} (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}})^2 d\mu_{ni} \leq \\ & \leq \sum_{i=1}^n Q_{ni}(r_{ni}(X_{ni}) \geq 1+\epsilon) + \sum_{i=1}^n P_{ni}(r_{ni}^{-1}(X_{ni}) \geq (1-\epsilon)^{-1}) \end{aligned}$$

and the equivalence of (3.5) and (3.6) to (3.7) is immediate.

Next we note that both (3.2), (3.3) and (3.4), (3.5), (3.6) imply $\{P_n^{(n)}\} \triangleleft \{Q_n^{(n)}\}$ (cf. corollary 1).

The remainder of the proof relies on the normal convergence theorem (cf. LOÈVE [5]). According to an equivalent form of this theorem (3.2) and (3.3) are equivalent to

$$(3.8) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n P_{ni}(|\log r_{ni}(X_{ni})| \geq \delta) = 0 \quad \text{for every } \delta > 0,$$

$$(3.9) \quad \lim_{\delta \downarrow 0} \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_{|\log r_{ni}| \leq \delta} (\log r_{ni}) dP_{ni} = -\frac{1}{2}\sigma^2,$$

$$(3.10) \quad \lim_{\delta \downarrow 0} \lim_{n \rightarrow \infty} \sum_{i=1}^n \left\{ \int_{|\log r_{ni}| \leq \delta} (\log r_{ni})^2 dP_{ni} - \left(\int_{|\log r_{ni}| \leq \delta} (\log r_{ni}) dP_{ni} \right)^2 \right\} = \sigma^2.$$

By the contiguity of $\{P_n^{(n)}\}$ and $\{Q_n^{(n)}\}$ and (2.1) the condition (3.8) is equivalent to (3.5) and (3.6) and hence to (3.7). Henceforth we assume (3.7), (3.8) and $\{P_n^{(n)}\} \triangleleft \{Q_n^{(n)}\}$. We still have to show that (3.4) is equivalent to (3.9) and (3.10).

Let $0 < \delta < 1$. For $|\log r_{ni}| \leq \delta$ we have the expansion

$$(3.11) \quad \begin{aligned} \log r_{ni} &= 2 \log\{1 + (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}}) p_{ni}^{-\frac{1}{2}}\} \\ &= 2(q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}}) p_{ni}^{-\frac{1}{2}} - (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}})^2 p_{ni}^{-1} (1 + \rho_{ni}\delta) \end{aligned}$$

with $|\rho_{ni}\delta| < 2\delta$. Thus

$$\begin{aligned} & \int_{|\log r_{ni}| \leq \delta} (\log r_{ni}) p_{ni} d\mu_{ni} = \\ &= -2 \int_{|\log r_{ni}| \leq \delta} (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}})^2 d\mu_{ni} + \int_{|\log r_{ni}| \leq \delta} (q_{ni} - p_{ni}) d\mu_{ni} + \\ & \quad - \int_{|\log r_{ni}| \leq \delta} \rho_{ni}\delta (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}})^2 d\mu_{ni}. \end{aligned}$$

Since by (3.7)

$$\lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^n \int_{|\log r_{ni}| \leq \delta} (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}})^2 d\mu_{ni} - \sum_{i=1}^n H^2(P_{ni}, Q_{ni}) \right\} = 0$$

and by (3.8), $\{P_n^{(n)}\} \nrightarrow \{Q_n^{(n)}\}$ and (2.1)

$$\sum_{i=1}^n \int_{|\log r_{ni}| \leq \delta} (q_{ni} - p_{ni}) d\mu_{ni} = - \sum_{i=1}^n \int_{|\log r_{ni}| > \delta} (q_{ni} - p_{ni}) d\mu_{ni} \rightarrow 0$$

for $n \rightarrow \infty$, we have

$$(3.12) \quad \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \left| \sum_{i=1}^n \int_{|\log r_{ni}| \leq \delta} (\log r_{ni}) dP_{ni} + 2 \sum_{i=1}^n H^2(P_{ni}, Q_{ni}) \right| \leq \\ \leq \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} 2\delta \sum_{i=1}^n H^2(P_{ni}, Q_{ni}) = 0,$$

where we have used (1.6). Similarly,

$$(3.13) \quad \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \sum_{i=1}^n \left\{ \int_{|\log r_{ni}| \leq \delta} (\log r_{ni}) dP_{ni} \right\}^2 \leq \\ \leq \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \delta \sum_{i=1}^n \left| \int_{|\log r_{ni}| \leq \delta} (\log r_{ni}) dP_{ni} \right| \leq \\ \leq \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \delta(2+2\delta) \sum_{i=1}^n H^2(P_{ni}, Q_{ni}) = 0.$$

Finally (3.11) implies that for $|\log r_{ni}| \leq \delta < 1$

$$(\log r_{ni})^2 = 4(q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}})^2 p_{ni}^{-1} + \bar{\rho}_{ni\delta} (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}})^2 p_{ni}^{-1}$$

with $|\bar{\rho}_{ni\delta}| < 10\delta$. Hence, in view of (3.7) and (1.6),

$$(3.14) \quad \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \left| \sum_{i=1}^n \int_{|\log r_{ni}| \leq \delta} (\log r_{ni})^2 dP_{ni} - 4 \sum_{i=1}^n H^2(P_{ni}, Q_{ni}) \right| = 0.$$

The equivalence of (3.4) to (3.9) and (3.10) is now an immediate consequence of (3.12), (3.13) and (3.14). The theorem is proved. \square

In the one sample case where, for each n , X_{n1}, \dots, X_{nn} are identically distributed, condition (3.3) is implied by (3.2) and theorem 2 slightly simplifies. This remains true in the k sample case ($k \geq 2$) provided all sample sizes tend to infinity.

The first part of the proof of theorem 2 also shows that the conditions (2.3) and (2.5) in corollary 1 may be replaced by the single condition

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \int_{|q_{ni} - p_{ni}| \geq c_n p_{ni}} (q_{ni}^{\frac{1}{2}} - p_{ni}^{\frac{1}{2}})^2 d\mu_{ni} = 0 \quad \text{whenever } c_n \rightarrow \infty.$$

The proof of theorem 2 could also be given in a more roundabout way. Introducing the r.v.'s

$$W_{ni} = 2\{q_{ni}(X_{ni})/p_{ni}(X_{ni})\}^{\frac{1}{2}} - 2, \quad i = 1, \dots, n; n = 1, 2, \dots,$$

one shows that $L(\sum_{i=1}^n W_{ni} | P_n^{(n)}) \xrightarrow{w} N(-\frac{1}{2}\sigma^2; \sigma^2)$ iff $L(\Lambda_n | P_n^{(n)}) \xrightarrow{w} N(-\frac{1}{2}\sigma^2; \sigma^2)$, provided the respective u.a.n. conditions are satisfied. It is then not difficult to prove that the weak convergence of $\sum_{i=1}^n W_{ni}$ and the u.a.n. condition on the summands are equivalent to (3.4) and (3.7). In this proof (3.7) appears as the Lindeberg condition in the central limit theorem applied to $\sum_{i=1}^n W_{ni}$.

The equivalence of both weak convergence results has first been proved by LECAM ([3],[4]). The initial assumptions $\lim_{n \rightarrow \infty} \sup_{1 \leq i \leq n} H^2(P_{ni}, Q_{ni}) = 0$ and $\limsup_{n \rightarrow \infty} \|P_n^{(n)} - Q_n^{(n)}\| < 1$ made by LeCam are not restrictive since they are implied by our condition (3.7) and the contiguity of $\{P_n^{(n)}\}$ and $\{Q_n^{(n)}\}$, respectively. One part of this proof is also contained in HÁJEK & ŠIDÁK [1].

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